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PLANE STRAIN DYNAMIC CRACK BIFURCATION

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Abstract—The problem of a stationary semi-infinite crack which bifurcates under dynamic loading is solved by constructing dual singular integral equations. The method relies on solving simpler problems which can be used with linear superposition to solve the more complex problem. An asymptotic approximation is made to allow a delay time in crack initiation to be admitted and this is extended to all times.

A number of fracture criteria are used to predict initiation of bifurcation and the results are shown to lie within the experimentally observed range.

INTRODUCTION

In fracture mechanics, the problem of a crack that is not straight has been singularly difficult to solve. For the static case, curving (and kinking) of a crack has been treated by asymptotic analysis. Independently, Banichuk[1] and Rice and Cotterell[2] (who gave a more complete interpretation of their solution) solved the problem to first order and Nemat-Nasser *et al.*[3] have considered higher order terms in the asymptotic expansion.

In dynamic fracture, it is only recently that correct solutions of the problem of a crack which kinks (or bifurcates) have been given. Burgers and Dempsey[4] gave some closed form results in anti-plane strain and Burgers[5] extended the range of these anti-plane results to all angles of kinking (and bifurcation) using a numerical scheme. Dempsey *et al.*[6] have used a scheme, initially used by Achenbach and Varatharajulu[7], to verify the results in[5]. In plane strain, Burgers[8] has given results for the kinking of a crack under various loadings, including stress wave loading. However, in all the dynamic results presented so far, the problems have been restricted to being self-similar in the radial coordinate and time.

The results for such loadings and the bifurcation case in plane strain are given below. The results are then extended asymptotically to allow for a delay time in the initiation of the bifurcation cracks. This expansion is valid for times long with respect to the delay time. To complete the analysis, these results will be matched with an estimate of the result for times short with respect to the delay time. As an estimate of accuracy, this combined asymptotic result for a bifurcation angle $\delta = 0$ is exactly the closed form result given by Freund[9] for a straight crack (taking into account that there are now two crack tips, instead of one).

THE SELF-SIMILAR BIFURCATION CRACK PROBLEM

The analysis herein follows that in [8] very closely and since many of the basic results are presented there, they will not be repeated. Only the superpositions which take into account the symmetry of the problem will be described as they were not used in [8]. All problems considered are solved in infinite isotropic linear elastic bodies with zero initial conditions.

The geometry of the bifurcated crack, at some time after bifurcation with a typical stress wave pattern is shown in Fig. 1. It should be noted that for the stresses to be self-similar the loading must be applied at the same instant the original semi-infinite crack bifurcates; that is at time $t = 0$. Also, to retain the self-similarity the bifurcated cracks must

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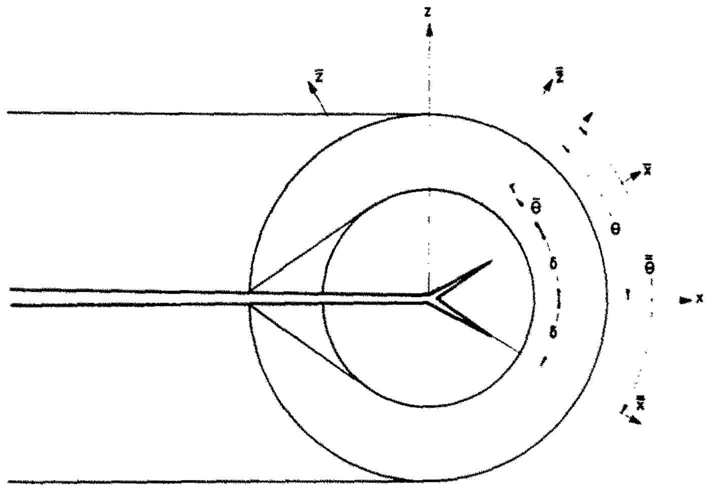

Fig. 1. Bifurcated crack under stress wave loading with typical wave front pattern.

both propagate at constant velocity. Here both crack tips are considered to propagate at the same velocity, v_{CT}, which is less than the Rayleigh wave speed, c_R.

The solution to the following two problems is required (see Fig. 2). We consider the case of the discontinuity in displacement perpendicular to its path first. At time $t = 0$, two displacement discontinuities normal to the direction of propagation are emitted from the crack tip at constant velocity w as shown, with the Burgers vector (displacement discontinuity) growing linearly with time. This problem is too difficult to solve as such so the solution is constructed using simpler problems.

In [8] the solution of a single normal displacement discontinuity with Burgers vector equal to Δt which suddenly appears at the origin of a semi-infinite body at time $t = 0$ with one end propagating out along the $\bar{x}$-axis ($\theta = \delta$) is given by $\Delta\underline{\sigma}^E(r/t, \bar{\theta}; w)$ (eqn 1.5) in [8]). (In [8] the normal displacement discontinuity is referred to as an edge dislocation. Since the reader is referred to [8] for the details of the calculations, the same notation as in [8] will be retained.) If we add to this problem a normal displacement discontinuity with

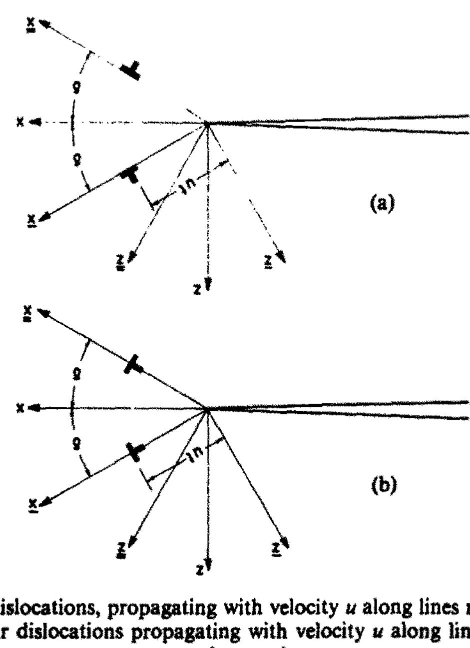

Fig. 2(a) Two edge dislocations, propagating with velocity u along lines making angles δ with the x-axis. (b) Two shear dislocations propagating with velocity u along lines making angles δ with the x-axis.

the same magnitude Δ, but propagating along the $\bar{x}$-axis ($\theta = -\delta$), the resulting stress field is symmetric about the x-axis. That is along the negative x-axis.

$$\sigma_{xz}(z=0, x<0) = 0,$$
$$\sigma_{xx}(z=0, x<0) = 2\Delta\sigma^{E}_{xx}(r/t, \bar{\theta} = \pi - \delta; w)$$

and

$$\sigma_{zz}(z=0, x<0) = 2\Delta\sigma^{E}_{zz}(r/t, \bar{\theta} = \pi - \delta; w). \tag{1}$$

To now construct the problem for the two dislocations propagating out of the tip of a traction free semi-infinite crack, the stress $\sigma_{zz}(z=0,\ x<0)$ has to be cancelled. (This problem involves less numerical calculation than the kinking crack case in [8].) This traction is cancelled using a normal point load which grows linearly with time and propagates at constant velocity out from the origin along the semi-infinite crack faces. The resulting stresses are given by

$$\underline{\sigma}^{EC}(r/t, \theta; u) = \underline{\sigma}^{E}(r/t, \theta; u) - \int_0^{c_L} 2\sigma^{E}_{zz}(v, \pi; u)\frac{\partial}{\partial v}\underline{\Sigma}^{NP}(r/t, \theta; v)\,\mathrm{d}v \tag{2}$$

where the notation in eqn (5.1) of Ref.[8] is used.

For the displacement discontinuity tangential to the direction of propagation, the arguments are very similar. The only point worth noting is that for this case, the Burgers vector of the displacement discontinuity propagating along $\theta = -\delta$ must be of opposite sign to that of the tangential displacement discontinuity along $\theta = \delta$ to obtain the same symmetry in the stress field.

By the symmetry of the problem we note that

$$\sigma_{\bar{z}\bar{z}}(r/t, \bar{\theta} = 0) = \sigma_{\bar{\bar{z}}\bar{\bar{z}}}(r/t, \bar{\bar{\theta}}) = 0$$

and

$$\sigma_{\bar{x}\bar{z}}(r/t, \bar{\theta} = 0) = -\sigma_{\bar{\bar{x}}\bar{\bar{z}}}(r/t, \bar{\bar{\theta}}) = 0. \tag{3}$$

Therefore, we only have to find a superposition of displacement discontinuities along $\theta = \delta$ (equivalently $\bar{\theta} = 0$) such that $\sigma_{\bar{z}\bar{z}}(\bar{\theta} = 0)$, $\sigma_{\bar{x}\bar{z}}(\bar{\theta} = 0)$ along the bifurcation crack are equal to the correct value. The conditions on $\theta = -\delta$ will be automatically satisfied by symmetry.

The result is two coupled Cauchy singular integral equations which can be solved numerically as in [8], with exactly the same assumptions made in the solution. The error in the stress intensity factor is expected to be of the same order as for the kinked crack case [8]. For the straight crack case, the error is less than 3% at crack tip velocities of 0.1, 0.3, 0.5, 0.7 and 0.9 c_R. Since there are no analytical solutions for bifurcated cracks available, a direct error analysis cannot be made. However, the method used follows that used in [5] for the corresponding anti-plane strain case where comparisons with certain analytical solutions could be made. Also, a semi analytical-numerical approach used in [6] verified the numerical solutions in [4, 5]. In this light, it is expected that the present solutions are of the same order of accuracy as in [8, 5], that is differing approx. 3% from the correct result

SYMMETRIC LOADING

Three loading cases will be considered. All are symmetric about $z = 0$ in keeping with the required symmetry of the problem. (The non-symmetric bifurcation case can be handled in a very similar manner but the enormous computer time requirements rule it out of this study.)

Case (a) has loading only on the bifurcated crack faces. The tractions on $\bar{z} = 0$, $0 < \bar{x} < v_{CT}t$ are

$$\sigma^{a}_{\bar{z}\bar{z}}(r/t, \bar{\theta} = 0) = \mu\Delta, \quad \sigma^{a}_{\bar{x}\bar{z}}(r/t, \bar{\theta} = 0) = 0. \tag{4}$$

Case (b) has constant loading only on the original semi-infinite crack faces. On $z = 0$, $x < 0$

$$\sigma^{b}_{zz}(r/t, \theta = 0) = \mu\Delta H(t), \quad \sigma^{b}_{xz}(r/t, \theta = 0) = 0 \tag{5}$$

where $H(t)$ is the Heaviside Step Function.

Case (c) is plane wave loading corresponding to a jump in the strain component ϵ_{zz} across the stress wave front, which is parallel to the semi-infinite crack. In eqns (7.3)–(7.6) of [8] this corresponds to $\alpha = 0$.

To consider the situation with shear loadings instead of the normal loads requires the stress field to be antisymmetric about $z = 0$. These cases are not formulated here. Note however that the results of case (a) for shear and normal loadings, and case (b) can be combined to give case (c), which is a longitudinal stress wave loading.

RESULTS FOR SELF-SIMILAR LOADING

The stress intensity factors for the above loadings are shown in Figs. 3–5 and the energy release rate per unit length crack advance G for the same loadings is given in Table 1. It is immediately noted that the results are significantly different to those for the kinked crack under the same loadings[8].

As an example for case (a), the mode II stress intensity factor K_{II} for the kinked crack case is essentially zero and the mode I stress intensity factor K_I is approximately constant, increasing slightly as δ approaches $0.5\,\pi$. The most interesting point in all loading cases is the behavior of K_{II}. For $\delta = 0$, $K_{II} = 0$ but for $\delta = 0.03125\,\pi$, K_{II} is negative. For larger δ and cases (b) and (c), K_{II} becomes positive with $K_{II} = 0$ between $\delta = 0.05\,\pi$ and $\delta = 0.1\,\pi$.

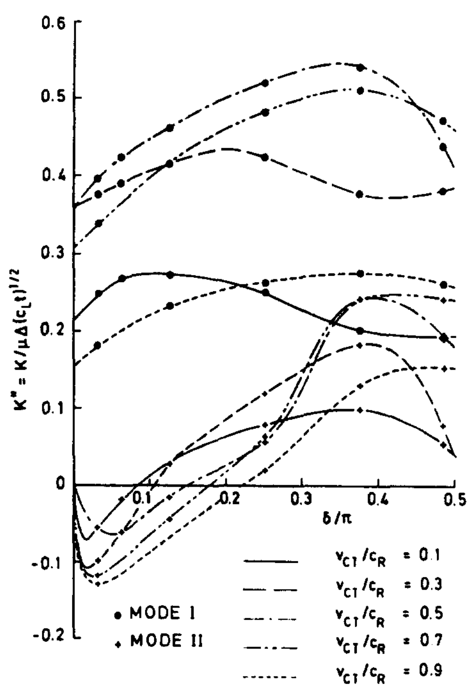

Fig. 3. Mode I and II stress intensity factors, K_I and K_{II}, as functions of crack tip speed v_{CT} and bifurcation angle δ for normal loading on the bifurcation crack faces.

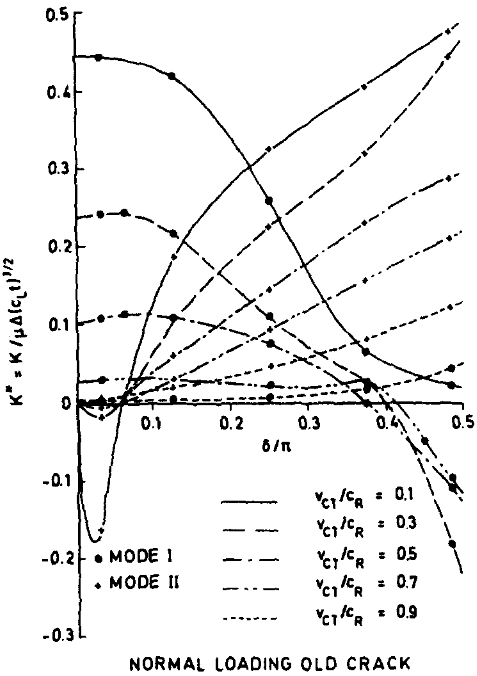

Fig. 4. Mode I and II stress intensity factors, K_I and K_{II}, as functions of crack tip speed v_{CT} and bifurcation angle δ for normal loading on the semi-infinite crack faces.

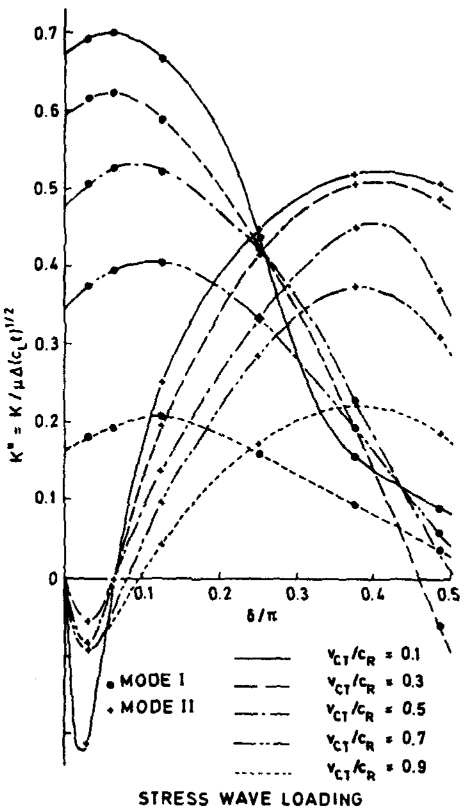

Fig. 5. Mode I and II stress intensity factors, K_I and K_{II}, as functions of crack speed v_{CT} and bifurcation angle δ for planar normal stress wave loading with the original wavefront parallel to the semi-infinite crack.

Table 1. $\mathcal{G}/((\mu\Delta)^2 c_L t)$ for loading cases (a)–(c)

v_{CT}/c_R	Loading Case	.03125	.125	.25	.375	.485
	a	.0485	.0559	.0522	.0376	.0308
.1	b	.1702	.1590	.1295	.1278	.1723
	c	.3963	.3861	.2964	.2210	.2002
	a	.1129	.1374	.1541	.1404	.1221
.3	b	.0466	.0455	.0565	.0800	.1804
	c	.3028	.3082	.2909	.2313	.1882
	a	.1479	.1946	.2482	.3138	.2062
.5	b	.0112	.0136	.0229	.0442	.0779
	c	.2881	.2667	.2563	.2187	.1150
	a	.1478	.2065	.2761	.3788	.3205
.7	b	.0011	.0025	.0088	.0249	.0529
	c	.1707	.2039	.2106	.1804	.0963
	a	.1058	.1419	.1678	.2166	.2081
.9	b	.0001	.0007	.0038	.0123	.0291
	c	.0934	.1104	.1135	.1004	.0594

For bifurcation (and crack curving) in static fracture it has been proposed that a suitable criterion to choose the crack path be that path on which $K_{II} = 0$. Using this criterion for the dynamic case, implies that bifurcation under loading cases (b) and (c) should occur at $\delta = 0$ or $0.05 \lesssim \delta/\pi \lesssim 0.1$. The latter range is within the range of experimental data, giving a range of included angle at bifurcation of 18–36°[16].

For case (b), at increasing crack tip velocities $K_I/(c_L t)^{1/2}$ has a maximum at increasing values of δ. In fact, for higher crack tip speeds, the maximum of $K/(c_L t)^{1/2}$ is closer to the point where $K_{II} = 0$. Therefore bifurcation may be expected in these cases. For v_{CT}/v_R slightly greater than 0.7, K_{II} is never zero and K_I is very small. The latter condition probably rules out crack propagation at speeds above $0.7\, c_R$ (as expected from experimental results).

For the criterion $K_{II} = 0$, it might be expected that for crack speeds less than $0.3\, c_R$ propagation straight ahead will be most favorable. Ravi-Chander[11] observed that for the situation essentially modeled by case (b) that the crack always propagated straight ahead for a short distance before bifurcating, at speeds approximately equal to $0.3\, c_R$, as predicted above.

For a planar stress wave (case c), the above criterion would predict bifurcation for all crack tip speeds with $0.0625 \lesssim \delta/\pi \lesssim 0.1$.

An alternative fracture criterion would be to consider a maximum for $(\mathcal{G}/t)$ but this has limited physical appeal. It is pointed out however that for case (b) $(\mathcal{G}/t)$ is largest at $\delta/\pi = 0.5$ whereas for case (a) a maximum occurs at large δ values as v_{CT} increases. It is interesting to note that this criterion also would not predict bifurcation for case (b) at a crack-tip velocity less than $\simeq 0.3\, c_R$, and does predict bifurcation in the observed range of angles of bifurcation.

APPROXIMATION FOR DELAY TIMES IN CRACK INITIATION

If the semi-infinite crack does not bifurcate at the instant of loading but at some later time, the problem loses its self-similar nature. However, Freund[9] has obtained the solution for a straight crack including a time delay and after making some observations of this remarkable result, it is possible to extend the above results to the case which includes a delay time, at least in an asymptotic sense.

We note that in posing the problem with inclusion of a delay time the boundary condition for loading is changed. For example, let a constant uniform loading be applied on the old crack faces at time $t = -\epsilon$ and the semi-infinite crack bifurcate at time $t = 0$.

The boundary conditions on the crack faces are then

$$\sigma_{zz}(x<0, z=0, t) = H(t+\epsilon)H(-x), \quad \sigma_{xz}(x<0, z=0, t) = 0. \tag{6}$$

We are interested in obtaining an asymptotic approximation to the stress for the time delay problem in terms of a perturbation about the self-similar problem with no time delay. For the self-similar problem, stress levels propagate out at constant velocity u from the original crack tip. Consider the loading term in the solution of the singular integral equations. With the time delay included, these are the stresses for step function loading at $t = -\epsilon$ on the old crack faces and can be written as $\underset{\sim}{\sigma}(\underset{\sim}{x}, t+\epsilon)$. How to expand this stress in terms of ϵ about a stress level propagating at velocity u along the $\bar{x}$-axis in the self-similar problem is most easily seen by introducing a coordinate system moving with the point of interest; that is let $\xi = \bar{x} - ut$ so that $\underset{\sim}{\sigma}(\bar{x}, 0, t+\epsilon) = \underset{\sim}{\sigma}(\xi, 0, t+\epsilon)$. Expanding this in terms of ϵ we obtain to second order in ϵ, for times $t \gg \epsilon > 0$,

$$\underset{\sim}{\sigma}(\xi, 0, t+\epsilon) = \underset{\sim}{\sigma}(\xi, 0, t) + \epsilon\frac{\partial}{\partial t}\underset{\sim}{\sigma}(\xi, 0, t) + 0(\epsilon^2). \tag{7}$$

If we consider the loading on the old crack faces which will cause a stress $(\partial\underset{\sim}{\sigma}/\partial t)(\xi, 0, t)$ we see that it is

$$\sigma_{zz}(x<0, 0, t) = \delta(t)H(-x) - uH(t)\delta(-x), \quad \sigma_{xz}(x<0, 0, t) = 0. \tag{8}$$

If we now consider the solution of the bifurcation problem with the loading in (6) we see that it can be obtained from the results presented above. The stress intensity factors for step function loading on the old crack, case (b), can be written as $K_{I,II} = K^*_{I,II}t^{1/2}$ where $K^*_{I,II}$ are constants. The time derivative of $K_{I,II}$ with respect to t is equivalent to taking the time derivative of the stress ahead of the crack tip, holding its position with respect to the crack tip fixed. That is, the time derivative is taken holding $\xi = \bar{x} - ut$ fixed where $u = v_{CT}$. This gives exactly the problem with boundary conditions (8).

The solution to the time delay problem for loading case (b) is then

$$K_{I,II} = K^*_{I,II}\left(t^{1/2} + \frac{\epsilon}{2}t^{-(1/2)}\right) + 0(\epsilon^2) \tag{9}$$

which to an error of $0(\epsilon^2)$ is $K^*_{I,II}(t+\epsilon)^{1/2}$. This is exactly what Freund[9] obtained as the exact result of the straight crack and $t > 0$. Therefore, we conclude to $0(\epsilon^2)$ for $t \gg \epsilon$ the stress intensity factor with a delay time ϵ is

$$K_{I,II} = K^*_{I,II}(t+\epsilon)^{1/2} \tag{10}$$

which is obtained from the self-similar results simply by replacing t by $(t+\epsilon)$.

To obtain a better understanding of the range of applicability of eqn (9) as a function of ϵ, the first term can be compared with Freund's result as given in eqn (10) for the straight crack case. For $\epsilon/t < 0.5$ the error is less than 10% which will be considered reasonable; i.e. the stress intensity calculated using eqn (9) will be accepted for $\epsilon/t < 0.5$. If the second approximation, given by eqn (10) is used the difference in Freund's result and this approximation is obviously zero for the straight crack case.

The difficulty is in obtaining a solution for $t \ll \epsilon$. If this solution could be obtained, the expansion for large t/ϵ could be matched and it would give a reasonably accurate prediction for all t. Unfortunately the actual short time results with a delay time seems to be as difficult to obtain as the complete solution.

To obtain an approximation to the short time solution, consider the following. The stress intensity factor can be written as $K_{I,II}(\delta, \epsilon; t)$ or alternatively as $\bar{K}_{I,II}(\delta, \epsilon; t+\epsilon)$. The latter form can be expanded in terms of $(t+\epsilon)$ so that

$$\tilde{K}_{I,II}(\delta,\epsilon;t+\epsilon) = \bar{K}_{I,II}(\delta,\epsilon)(t+\epsilon)^{1/2} + \text{higher order terms in } (t+\epsilon). \tag{11}$$

Powers of $(t+\epsilon)$ less than 1/2 are ruled out since in the limiting case, $\delta = 0$ or $\epsilon = 0$, they give the incorrect result.

$\bar{K}_{I,II}(\delta,\epsilon)$ can now be expanded in terms of ϵ

$$\bar{K}_{I,II}(\delta,\epsilon) = \bar{K}_{I,II}(\delta,\epsilon=0) + \text{higher order terms in } \epsilon. \tag{12}$$

Therefore

$$K_{I,II}(\delta,\epsilon;t) \simeq \bar{K}_{I,II}(\delta,\epsilon=0)(t+\epsilon)^{1/2} + \text{high order terms.} \tag{13}$$

For $\epsilon = 0$, it has already been shown that

$$K_{I,II}(\delta,\epsilon=0;t) = K^*_{I,II}(\delta)t^{1/2} \tag{14}$$

so we conclude $\bar{K}_{I,II}(\delta,\epsilon=0) = K^*_{I,II}(\delta)$.

The stress intensity factor can now be written as

$$K_{I,II}(\delta,\epsilon;t) = K^*_{I,II}(\delta)(t+\epsilon)^{1/2} + \text{higher order terms} \tag{15}$$

where the expansion is valid then $(t+\epsilon) \ll 1$.

We now see that the 1st order term of the long time expansion when $(t+\epsilon) \gg 1$ and the short time expansion when $(t+\epsilon) \ll 1$, are the same. We can then conclude that the uniform asymptotic expansion to first order for all $(t+\epsilon)$ is as given by eqn (10). It has been assumed here that the crack tip velocity is constant and although it should be possible to generalize it to varying crack tip speeds, the corresponding term to $K^*_{I,II}(\delta)$ is unknown. This would require a detailed numerical solution which is currently not available.

DISCUSSION

With the inclusion of a delay time, the above fracture criterion make a great deal more sense physically. A critical stress intensity factor now has a fixed magnitude. Since the form of the stress intensity given in eqn (15) is the same as for the self-similar case discussed earlier, the conclusions will not be repeated here, except to note that the predicted bifurcation angles for loading cases (b) and (c) are within the observed range of bifurcation angles [16], though at the low end.

An energy based fracture criterion, initially proposed by Eshelby[12] and later considered by Freund[13], can be used to look at the initiation of bifurcation. The criterion is based on the assumption that the energy release rate at initiation of fracture is a material parameter. For static fracture under small scale yielding conditions this is well established, but for dynamic fracture it is not clear that this is a suitable criterion if it is accepted that the crack can start propagating at a non-zero velocity. However, assuming that an energy based criterion is suitable, the condition of crack growth for any crack tip is

$$\mathcal{G}^{\text{applied}} = \mathcal{G}^{\text{mat}} \tag{16}$$

where $\mathcal{G}^{\text{applied}}$ can be obtained from the above results and $\mathcal{G}^{\text{mat}}$ is the fracture toughness of the material (and may be a function of velocity of the crack tip and position). Since it does not seem reasonable that $\mathcal{G}^{\text{mat}}$ be an explicit function of time, this criterion is not suitable for considering crack growth after initiation for the above calculations, as pointed out in[15].

The question now is, under the applied loading, will the crack have a tendency to propagate straight ahead or bifurcate? Consider $\mathcal{G}^{\text{mat}}$ to be a constant for simplicity and let the crack initiate at some time ϵ after loading. The values in Table 1 are of the form

$\mathscr{G} = \mathscr{G}^*(v_{CT}, \delta)c_L t(\mu\Delta)^2$. If the discussion above on how to include a delay time is used, at initiation of crack growth

$$\mathscr{G}^{\text{applied}} = \mathscr{G}^*(v_{CT}, \delta)c_L\epsilon(\mu\Delta)^2 = \mathscr{G}^{\text{mat}} \tag{17}$$

Obviously for the loadings considered $\epsilon > 0$.

There is no way to fix the value of ϵ at present. If we consider $\mathscr{G}^*$ to be a function of v_{CT} and δ and look for maximum values, in both loading cases (b) and (c) there is a local maximum for $v_{CT} = 0$ and $\delta = 0$. Therefore we might expect no bifurcation and quasi-static crack growth. The applied energy release rate is growing linearly with time, however, and to satisfy the fracture criterion it will have to decrease this rate of growth. The only way this can be done without bifurcation is for the crack to accelerate. It is possible that the very rapid acceleration phase of crack growth seen by Kalthoff *et al.*[14] is not just an experimental error but part of this instability of crack growth. At this point a combination of bifurcation and acceleration effects cannot be ruled out.

Alternatively if crack initiation is delayed in any way (slight blunting of the crack tip for example), the crack tip region will be in a super critical state immediately after initiation. Another way of obtaining this condition is if $\mathscr{G}^{\text{mat}}$ for initiation and quasi-static growth is greater than that for dynamic growth. As soon as the crack starts growing it must do so at a velocity greater than zero. This introduces the possibility that more favorable paths for the release of energy than a straight path can be found. For example, for loading case (b) with crack tip speeds less than approx. $0.3\,c_R$ the crack will most likely grow straight ahead (initially) as observed by Ravi-Chander[11]. For higher values of v_{CT}, $\delta = \pi/2$ seems the most likely angle of bifurcation but these crack tip speeds are higher than commonly observed.

As suggested by Freund[13], this can lead to the crack tip searching for new paths. At v_{CT} in the range $0.3\,c_r$ to $0.5\,c_r$ the crack may try to bifurcate but because the energy requirements for going straight ahead are so close to that for bifurcation, the crack tip will make repeated attempts at bifurcation without succeeding. This will absorb energy and effectively increase the required energy release rate to a level where continued crack growth can occur smoothly.

For loading case (c), $\mathscr{G}^*$ is again a maximum at $v_{CT} = 0$, $\delta = 0$. However, if the crack tip region becomes supercritical in any way, bifurcation becomes a possibility for $v_{CT} \gtrsim 0.3\,c_R$, with δ approximately in the range of $0.125\,\pi$ to $0.25\,\pi$. That is, at crack tip speeds observed in experiments bifurcation is predicted at angles within the range observed.

The inclusion of a delay time allows something reasonably definite to be said about initiation of crack growth. In this case, the above criterion are valid for initiation, although they have the same limitation when extended to times after initiation. It is unfortunately not yet clear from experimental results what is an appropriate criterion for bifurcation since no measurements of the stress intensity factors after bifurcation are available.

The next step in the analysis is to allow crack propagation before bifurcation. This problem, however is a great deal more difficult to solve but it must be attempted if the bifurcation question is to be resolved.

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